DISTANCES TO SPACES OF BAIRE ONE FUNCTIONS

C. ANGOSTO, B. CASCALES AND I. NAMIOKA

ABSTRACT. Given a metric space X and a Banach space $(E, \|\cdot\|)$ we use an index of σ-fragmentability for maps $f \in E^X$ to estimate the distance of f to the space $B_1(X, E)$ of Baire one functions from X into $(E, \|\cdot\|)$. When X is Polish we use our estimations for these distances to give a quantitative version of the well known Rosenthal's result stating that in $B_1(X, \mathbb{R})$ the pointwise relatively countably compact sets are pointwise compact. We also obtain a quantitative version of a Srivatsa's result that states that whenever X is metric any weakly continuous function $f \in E^X$ belongs to $B_1(X, E)$: our result here says that for an arbitrary $f \in E^X$ we have

$$d(f, B_1(X, E)) \leq 2 \sup_{x^* \in B_{E^*}} \operatorname{osc}(x^* \circ f),$$

where $\operatorname{osc}(x^* \circ f)$ stands for the supremum of the oscillations of $x^* \circ f$ at all points $x \in X$. As a consequence of the above we prove that for functions in two variables $f : X \times K \to \mathbb{R}$, X complete metric and K compact, there exists a G_δ-dense set $D \subset X$ such that the oscillation of f at each $(x, k) \in D \times K$ is bounded by the oscillations of the *partial* functions f_x and f^k. A representative result in this direction, that we prove using games, is the following: if X is a σ-β-unfavorable space and K is a compact space, then there exists a dense G_δ-subset D of X such that, for each $(y, k) \in D \times K$,

$$\operatorname{osc}(f, (y, k)) \leq 6 \sup_{x \in X} \operatorname{osc}(f_x) + 8 \sup_{k \in K} \operatorname{osc}(f^k).$$

When the right hand side of the above inequality is zero we are dealing with separately continuous functions $f : X \times K \to \mathbb{R}$ and we obtain as a particular case some well-known results obtained by the third named author in the mid 1970's.

1. INTRODUCTION

Many results in mathematics are *qualitative* and some other results are of *quantitative* nature. For example, one consequence of Hahn-Banach theorem is that, in a locally convex space E a compact convex set A disjoint from a closed convex set B can be separated by an element x^* of the dual E^*. Although we might think of this result as a result of *qualitative* nature, we know that its true power is behind its *quantitative* disguise: in the above situation $x^* \in E^*$ can be chosen to satisfy

$$\sup_{a \in A} x^*(a) \leq \alpha < \alpha + \varepsilon \leq \inf_{b \in B} x^*(b),$$

for some $\alpha \in \mathbb{R}$ and $\varepsilon > 0$.

Recently several *quantitative* counterparts for classical results such as Krein-Šmulian, Eberlein-Šmulian, Grothendieck, etc., have been proved. They strengthen the original theorems and lead to new problems and applications in topology and analysis. See, for

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instance, [1, 4, 8, 9, 10, 11, 12]. This paper goes along the same line, and it is organized as follows.

In Section 2 we use the concept of σ-fragmented maps to give a distance between a given function on a metric space (or more generally a perfectly paracompact space) X into a Banach space E to the space $B_1(X, E)$. Our Theorem 2.5 properly extends a result in [11].

In Section 3 we use the results of the previous section to study compactness relative to $B_1(X)$. Proposition 3.1 and its Corollary represent a quantitative version of the celebrated Rosenthal's result that in $B_1(X)$ the pointwise relatively countably compact sets are pointwise compact when X is Polish.

Section 4 is a technical bridge to Section 5 where we offer, in Theorem 4.1, a quantitative version of the following Mazur's result: Suppose X is a countably compact space and $(f_n)_n$ is a uniformly bounded sequence of continuous functions defined on X converging pointwise to a continuous function f. Then there exists a sequence $(g_n)_n$ in $\operatorname{conv}\{f_n : n \in \mathbb{N}\}$ that is $\|\cdot\|_\infty$-convergent to f.

In Section 5 we use Theorem 4.1 to establish that for a function $F : X \to \mathbb{R}^K$ (X metric space and K a compact) we have

$$d(F, B_1(X, C(K))) \leq \frac{3}{2} \sup_{x \in X} \operatorname{osc}((F(x)) + 2 \sup_{k \in K} \operatorname{osc}(\pi_k \circ F)$$

where π_k is the map : $C(K) \to \mathbb{R}$ given by $h \mapsto h(x)$. As application of the above we obtain the quantitative version of Srivatsa's result as presented in the abstract, its relative Corollary 5.3 and a first Namioka's type result, Corollary 5.7, that states that if X is a complete metric space, K is a compact space and $f : X \times K \to \mathbb{R}$ is any function, then there exists a dense $\mathcal{G}_\delta$ set $D \subset X$ such that for every $(x, k) \in D \times K$

$$\operatorname{osc}(f, (x, k)) \leq 7 \sup_{x \in X} \operatorname{osc}(f_x) + 8 \sup_{k \in K} \operatorname{osc}(f^k). \tag{QN}$$

Section 6, and last one, is devoted to prove that using *topological games* inequality (QN) can be extended to a wider class of spaces X and also sharpened. Indeed, amongst other things we prove that (QN) still holds for σ-β-unfavorable spaces X and that the coefficient 7 in (QN) can be replaced by 6, see Theorem 6.1. Furthermore, if we add the hypothesis that X is normal, then even the second coefficient 8 in (QN) can be replaced by 7, see Corollary 6.3. Finally, we prove (QN)-type results with better coefficients when X is a Baire space and K belongs to a certain class of compact spaces (including Corson and even Valdivia compact spaces) defined by a topological game, see Theorem 6.9 and their corollaries.

Notation and terminology: We denote by letters $T, X, Y, \ldots$ sets or completely regular topological spaces, and (Z, d) (or simply Z if d is tacitly assumed) is a metric space . The space Z^X is equipped with the product topology τ_p. In Z^X we also consider the *standard supremum metric*, which is also (abusively) denoted by d and allowed to take the value $+\infty$, i.e.,

$$d(f, g) = \sup\{d(f(x), g(x)) : x \in X\}$$

for functions $f, g : X \to Z$. It is always possible to replace the original metric d in (Z, d) by a bounded one without changing the uniform structure of Z and thus providing us with a real-valued uniform metric on Z^X; nonetheless we prefer to use the original metric of (Z, d) with the usual arithmetical convention for $[0, +\infty]$. We let $C(X, Z)$ denote the space of all Z-valued continuous functions on X, and let $B_1(X, Z)$ denote the space of all Z-valued functions of the first Baire class (Baire one functions), i.e. pointwise limits of Z-valued continuous functions. When $Z = \mathbb{R}$, we write, as usual, $C(X)$ and $B_1(X)$ for $C(X, \mathbb{R})$ and $B_1(X, \mathbb{R})$ respectively.

For non-empty subsets A and B of a metric space (Z, d), we consider the *usual distance* between A and B given by

$$d(A, B) = \inf\{d(a, b) : a \in A, b \in B\},$$

and the *Hausdorff non-symmetrized distance* from A to B defined by

$$\hat{d}(A, B) = \sup\{d(a, B) : a \in A\}.$$

If $\emptyset \neq A \subset Z$ we write $\operatorname{diam}(A) := \sup\{d(x, y) : x, y \in A\}$. In this paper $(E, \|\cdot\|)$ is a Banach space (or simply E if $\|\cdot\|$ is tacitly assumed). Finally, B_E stands for the closed unit ball in E, E^* for the dual space of E and E^{**} for the bidual space of E; w is the weak topology of a Banach space and w^* is the weak* topology in the dual.

2. INDEX OF σ-FRAGMENTABILITY AND DISTANCE TO $B_1(X, E)$

Recall that for a given $\varepsilon > 0$, a metric space-valued function $f : X \to (Z, d)$ is ε-fragmented if for each non-empty subset $F \subset X$ there exists an open subset $U \subset X$ such that $U \cap F \neq \emptyset$ and $\operatorname{diam}(f(U \cap F)) \leq \varepsilon$. Given $\varepsilon > 0$, we say that f is ε-σ-fragmented by *closed sets* if there is a countable closed covering $(X_n)_n$ of X such that $f|_{X_n}$ is ε-fragmented for each $n \in \mathbb{N}$.

If U is open in X and $A \subset X$, then $U \cap \overline{A} \neq \emptyset$ if and only if $U \cap A \neq \emptyset$; it follows that the function f is ε-fragmented if for each closed set $F \subset X$ there exists an open subset $U \subset X$ such that $U \cap F \neq \emptyset$ and $\operatorname{diam} f(U \cap F) \leq \varepsilon$.

Definition 1. Let X be a topological space, (Z, d) a metric space and $f \in Z^X$ a function. We define:

$$\operatorname{frag}(f) := \inf\{\varepsilon > 0 : f \text{ is } \varepsilon\text{-fragmented}\},$$
$$\sigma\text{-}\operatorname{frag}_c(f) := \inf\{\varepsilon > 0 : f \text{ is } \varepsilon\text{-}\sigma\text{-fragmented by closed sets}\},$$

where by definition, $\inf \emptyset = +\infty$.

Using the above, the usual notion of (σ-) fragmentability [15, p. 248] can be defined as follows:

(i) f *is fragmented* if and only if $\operatorname{frag}(f) = 0$.
(ii) f *is σ-fragmented by closed sets* if and only if σ-$\operatorname{frag}_c(f) = 0$.

Theorem 2.1. *Let X be a topological space and (Z, d) a metric space. If $f \in Z^X$ then the following inequality holds*

$$\sigma\text{-}\operatorname{frag}_c(f) \leq \operatorname{frag}(f).$$

If moreover X is hereditarily Baire, then

$$\sigma\text{-}\operatorname{frag}_c(f) = \operatorname{frag}(f).$$

Proof. The inequality follows from the definition. Suppose now that X is hereditarily Baire. We have to prove that if $+\infty > \varepsilon > \sigma$-$\operatorname{frag}_c(f)$, then f is ε-fragmented. Let C be a closed subset of X. Fix $\varepsilon > \sigma$-$\operatorname{frag}_c(f)$. Then there is a sequence $(X_n)_n$ of closed sets covering X such that $f_{|X_n}$ is ε-fragmented for each $n \in \mathbb{N}$. Put $H_n = X_n \cap C$. Then, the sequence of closed sets $(H_n)_n$ covers C, and, since C is a Baire space, there exist an $n \in \mathbb{N}$ and a open set $U \subset X$ such that $\emptyset \neq U' = U \cap C \subset H_n \subset X_n$. Since

$f|_{X_n}$ is ε-fragmented, there exists an open subset V of X such that $V \cap U' \neq \emptyset$ and $\operatorname{diam} f(V \cap U') \leq \varepsilon$. Put $W = V \cap U$. Then W is an open subset of X and

$$W \cap C = V \cap U \cap C = V \cap U' \neq \emptyset.$$

Hence $\operatorname{diam} f(W \cap C) \leq \varepsilon$. Therefore f is ε-fragmented. □

A family $\mathcal{A}$ of subsets of a topological space X is said to be *discrete* if each $x \in X$ has a neighborhood in X that intersects at most one member of $\mathcal{A}$. In order to study the relation between σ-$\mathrm{frag_c}(f)$ and distances to $B_1(X, E)$, we use the following notions.

Definition 2. Let X be a topological space.

(i) An indexed family $\{X_i : i \in I\}$ of subsets of X is said to be *discretely σ-decomposable* (*d.σ.d*) if for each $i \in I$ we have $X_i = \bigcup\{X_{i,n} : n \in \mathbb{N}\}$, where the family $\{X_{i,n} : i \in I\}$ is discrete for each $n \in \mathbb{N}$.

(ii) An indexed family $\{X_i : i \in I\}$ of subsets of X is said to be a *good partition of* X if it is a *d.σ.d* family of $\mathcal{F}_\sigma$-subsets of X such that $X_i \cap X_j = \emptyset$ if $i \neq j$ and $X = \bigcup_{i \in I} X_i$.

Remark 2.2. Let X be a topological space then

(i) If $\{X_i : i \in H\}$ is a good partition of X, and if for each $i \in I$, $\{Y^i_j : j \in J_i\}$ is a good partition relative to the subspace X_i, then it is easy to check that $\{Y^i_j : i \in I, j \in J_i\}$ is a good partition of X (see [14, Lemma 5]).

(ii) If $\{X_n : n \in \mathbb{N}\}$ is a countable cover of X by closed sets and each open subset of X is a $\mathcal{F}_\sigma$ set, then $\{Y_n : n \in \mathbb{N}\}$ is a good partition of X where $Y_1 = X_1$ and for $n > 1$, $Y_n = X_n \setminus \bigcup_{m=1}^{n-1} X_m$. Note now that any countable partition of sets is *d.σ.d.*

The following are the results needed for the proof of Theorem 2.5 in which we use ideas of [15].

Proposition 2.3 ([15, Proposition 2])**.** *Let X be a metric space and $\{G_\gamma : \gamma < \Gamma\}$ a transfinite sequence of open sets covering X. If $F_\gamma = G_\gamma \setminus \bigcup_{\xi<\gamma} G_\xi$, then $\{F_\gamma : \gamma < \Gamma\}$ is a good partition of X.*

Lemma 1. *Let Y be a metric space, E a convex subset of a Banach space, $\varepsilon > 0$ and $f : Y \to E$ an ε-fragmented function. Then there exist a function $h : Y \to E$ which is constant on each set of a good partition of Y such that*

$$\|f(y) - h(y)\| \leq \varepsilon \text{ for all } y \in Y.$$

Moreover, if $E = \mathbb{R}$ then the h above can chosen to satisfy

$$\|f(y) - h(y)\| \leq \varepsilon/2 \text{ for all } y \in Y.$$

Proof. Since f is ε-fragmented, there exist an ordinal Γ and an open cover $\{G_\gamma : \gamma < \Gamma\}$ of Y such that $F_\gamma \neq \emptyset$ and $\operatorname{diam} f(F_\gamma) \leq \varepsilon$, where $F_0 = G_0$ and, for $0 < \gamma < \Gamma$, $F_\gamma = G_\gamma \setminus \bigcup_{\xi<\gamma} G_\xi$. By Proposition 2.3, $\{F_\gamma : \gamma < \Gamma\}$ is a good partition of Y. If $E = \mathbb{R}$, choose t_γ the middle point of $\operatorname{conv} f(F_\gamma)$ and if $E \neq \mathbb{R}$ choose t_γ an arbitrary point of $f(F_\gamma)$ for $\gamma < \Gamma$. Define now $h : Y \to E$ by $h(y) = t_\gamma$ if $y \in F_\gamma$. Clearly h is a constant function on each set of a good partition of Y that satisfies $d(f, h) \leq \varepsilon$, and if $E = \mathbb{R}$, then $d(f, h) \leq \varepsilon/2$. □

Proposition 2.4 ([15, Proposition 3])**.** *Let X be a metric space and E a convex subset of a Banach space. If $f : X \to E$ is constant on each set of a good partition, then $f \in B_1(X, E)$.*

Theorem 2.5. *If X is a metric space, E a Banach space and if $f \in E^X$ then*

$$\frac{1}{2}\,\sigma\text{-frag}_{\mathrm{c}}(f) \leq d(f, B_1(X,E)) \leq \sigma\text{-frag}_{\mathrm{c}}(f). \tag{2.1}$$

In the case $E = \mathbb{R}$ we have the equality

$$d(f, B_1(X)) = \frac{1}{2}\,\sigma\text{-frag}_{\mathrm{c}}(f). \tag{2.2}$$

Proof. We prove the first inequality in (2.1). If $d(f, B_1(X,E)) = +\infty$ the inequality holds. So suppose that $d(f, B_1(X,E))$ is finite. Fix $\alpha > d(f, B_1(X,E))$ and take $g \in B_1(X,E)$ with

$$\|f(x) - g(x)\| < \alpha, \text{ for every } x \in X. \tag{2.3}$$

Pick a sequence $(g_n)_n$ in $C(X,E)$ such that $\lim_n g_n(x) = g(x)$ for every $x \in X$. Fix $\varepsilon > 0$ and for each $n \in \mathbb{N}$ let us define

$$X_n := \bigcap_{m \geq n} \{x \in X : \|g_n(x) - g_m(x)\| \leq \frac{\varepsilon}{3}\}.$$

It is clear that each X_n is closed and that $X = \bigcup_n X_n$. On the other hand, for every $x \in X_n$ we have that

$$\|g_n(x) - g(x)\| \leq \frac{\varepsilon}{3}. \tag{2.4}$$

Since g_n is continuous, for each $a \in X_n$ we can take an open neighborhood $V_a \subset X$ of a such that $\operatorname{diam} g_n(V_a) < \frac{\varepsilon}{3}$. Hence (2.3) and (2.4) allow us to conclude that $\operatorname{diam} f(V_a \cap X_n) < 2\alpha + \varepsilon$. This last inequality says that $\sigma\text{-frag}_{\mathrm{c}}(f) \leq 2\alpha + \varepsilon$ for every $\varepsilon > 0$. Since $\alpha > d(f, B_1(X,E))$ is arbitrary we conclude that $\sigma\text{-frag}_{\mathrm{c}}(f) \leq 2d(f, B_1(X,E))$ and this part of the proof is over.

We prove now the last inequality in (2.1). Since the inequality holds when $\sigma\text{-frag}_{\mathrm{c}}(f) = +\infty$, we can suppose that $\sigma\text{-frag}_{\mathrm{c}}(f)$ is finite. Given $\varepsilon > \sigma\text{-frag}_{\mathrm{c}}(f)$, there exists a countable closed cover $\{X_n : n \in \mathbb{N}\}$ of X such that $f|_{X_n}$ is ε-fragmented. By Remark 2.2 (ii), $\{Y_n : n \in \mathbb{N}\}$ is a good partition of X, where $Y_1 = X_1$, and for $n > 1$, $Y_n = X_n \setminus \bigcup_{m=1}^{n-1} X_m$. By Lemma 1 applied to each Y_n, there exists a function $g_n : Y_n \to E$ that is constant on each set of a good partition of Y_n such that if $x \in Y_n$ then $d(g_n(x), f(x)) \leq \varepsilon$ and $d(g_n(x), f(x)) \leq \varepsilon/2$ if $E = \mathbb{R}$. Define $g : X \to E$ as

$$g(x) = g_n(x) \text{ if } x \in Y_n.$$

By Remark 2.2 (i), g is constant on each set of a good partition of X so by Proposition 2.4, $g \in B_1(X,E)$. Clearly, $d(f,h) \leq \varepsilon$ in general so the second inequality in (2.1) holds, and if $E = \mathbb{R}$, then we have $d(f,h) \leq \varepsilon/2$ and the equality (2.2) is proved. □

Combining Theorem 2.5 and Theorem 2.1 we get the following corollary.

Corollary 2.6. *If X is a hereditarily Baire metric space and $f \in \mathbb{R}^X$, then*

$$d(f, B_1(X)) = \frac{1}{2}\operatorname{frag}(f).$$

We note that the corollary above extends [11, Proposition 6.4.], where 2.6 is proved only for X Polish.

Remark 2.7. We stress that according to the references [15] and [14], Proposition 2.3 holds for a perfectly paracompact space, i.e., a paracompact space for which open sets are $\mathcal{F}_\sigma$ sets. Therefore, Theorem 2.5 and its corollary are also true when 'metric' is replaced by 'perfectly paracompact' .

3. Quantitative difference between countable compactness and compactness in $B_1(X,E)$

We use the following notation in this section. Let T be a topological space. For a subset A of T, $A^{\mathbb{N}}$ is considered as the set of all sequences in A and the set of all cluster points in T of a sequence $\varphi \in A^{\mathbb{N}}$ is denoted by $\operatorname{clust}(\varphi)$. Recall that $\operatorname{clust}(\varphi)$ is a closed subset of T and it can be expressed as

$$\operatorname{clust}(\varphi) = \bigcap_{n\in\mathbb{N}} \overline{\{\varphi(m) : m \geq n\}}.$$

Proposition 3.1. *Let X be a separable metric space, (Z,d) a metric space and H a pointwise relatively compact subset of (Z^X,τ_p). Then (closures are taken relative to τ_p),*

$$\sup_{f\in\overline{H}} \operatorname{frag}(f) = \sup_{\varphi\in H^{\mathbb{N}}} \inf\{\operatorname{frag}(f) : f \in \operatorname{clust}(\varphi)\}. \tag{3.1}$$

Proof. Let α be the right hand side of (3.1). Clearly

$$\beta := \sup_{f\in\overline{H}} \operatorname{frag}(f) \geq \alpha.$$

If $\beta = 0$ we are done. Otherwise, the equality (3.1) will be established if we prove that each time $\beta > \varepsilon > 0$ we also have $\alpha \geq \varepsilon$. Assume $\beta > \varepsilon > 0$ and pick $f \in \overline{H}$ such that $\operatorname{frag}(f) > \varepsilon$. Then there exists a non-empty subset $F \subset X$ such that $\operatorname{diam} f(F\cap U) > \varepsilon$ for each open set $U \subset X$ with $U \cap F \neq \emptyset$. Let us fix $\{U_n : n \in \mathbb{N}\}$ a basis for the topology in X and write $B := \{n \in \mathbb{N} : U_n \cap F \neq \emptyset\}$. For every $n \in B$ we can choose $x_n, y_n \in U_n \cap F$ such that $d(f(x_n), f(y_n)) > \varepsilon$. Let us write $C := \{x_n : n \in B\} \cup \{y_n : n \in B\}$. Since $C \subset X$ is countable and $f \in \overline{H}$ there exists a sequence $\varphi \in H^{\mathbb{N}}$ such that $\lim_n \varphi(n)(x) = f(x)$ for every $x \in C$. Therefore, if g is an arbitrary τ_p-cluster point of φ then $g|_C = f|_C$ and in particular we have that

$$d(g(x_n), g(y_n)) > \varepsilon, \text{ for every } n \in B. \tag{3.2}$$

If U is an open set such that $U \cap C \neq \emptyset$ then there exist $n \in \mathbb{N}$ such that $\emptyset \neq U_n \cap C \subset U \cap F$. Hence, $n \in B$ and since $x_n, y_n \in U \cap C$ we conclude

$$\operatorname{diam} g(U\cap C) \geq d(g(x_n), g(y_n)) \overset{(3.2)}{>} \varepsilon.$$

We have proved that

$$\inf\{\operatorname{frag}(g) : g \in \operatorname{clust}(\varphi)\} \geq \varepsilon$$

and therefore $\alpha \geq \varepsilon$ so the proof is complete. □

If X is a topological space, (Z,d) a metric space and H a relatively compact subset of the space (Z^X,τ_p) we define

$$\operatorname{ck}(H) := \sup_{\varphi\in H^{\mathbb{N}}} d(\operatorname{clust}(\varphi), B_1(X,Z)). \tag{3.3}$$

Note that if H is a relatively countably compact subset of $(B_1(X,Z),\tau_p)$, then $\operatorname{ck}(H) = 0$. Combining Proposition 3.1, Theorem 2.5 and Theorem 2.1, we get the following result. The particular case of $\operatorname{ck}(H) = 0$ and $E = \mathbb{R}$ is the classic result due to Rosenthal [18].

Corollary 3.2. *Let X be a Polish space, E a Banach space and H a τ_p-relatively compact subset of E^X. Then*

$$\operatorname{ck}(H) \leq \hat{d}(\overline{H}^{E^X}, B_1(X,E)) \leq 2\operatorname{ck}(H).$$

In the particular case when $E = \mathbb{R}$ we have

$$\hat{d}(\overline{H}^{\mathbb{R}^X}, B_1(X)) = \mathrm{ck}(H).$$

4. A QUANTITATIVE VERSION OF A THEOREM BY MAZUR

In this section, we follow the notation of [21, Section 1.3]. Our goal in this section is to prove Theorem 4.1 which is used in the proof of Theorem 5.2. $\mathbb{N}^{(\mathbb{N})}$ denotes the set of finite subsets of $\mathbb{N}$.

Definition 3. A *convex mean* on $\mathbb{N}$ is a function $\mu : \mathbb{N} \to [0,1]$ such that

(i) $\sum_{i=1}^{\infty} \mu(i) = 1$
(ii) $\mathrm{Supp}(\mu) = \{i \in \mathbb{N} : \mu(i) > 0\}$ is finite.

For $F \subset \mathbb{N}$, let $\mu(F) = \sum_{i \in F} \mu(i)$. $\mathbf{M}$ denote the set of all convex means on $\mathbb{N}$.

Definition 4. Suppose $\mathcal{F} \subset \mathbb{N}^{(\mathbb{N})}$. Then $\mathcal{F}$ is *well-founded* if there is no infinite increasing sequence $(n_i)_i \subset \mathbb{N}$ such that for each $k \in \mathbb{N}$, there exists $S_k \in \mathcal{F}$ such that $\{n_1, \ldots, n_k\} \subset S_k$.

Lemma 2 (V. Pták, [21, p. 13]). *If $\mathcal{F}$ is a well-founded family of finite subsets of $\mathbb{N}$, then for each $\varepsilon > 0$ there is a $\mu \in \mathbf{M}$, such that $\mu(F) < \varepsilon$ for all $F \in \mathcal{F}$.*

Theorem 4.1 (Quantitative Mazur Theorem)**.** *Suppose that X is a countably compact space, $a > 0$ and $(f_n)_n \subset \mathbb{R}^X$ is a uniformly bounded sequence such that for each $x \in X$ there exists an n_x such that*

$$\text{if } n > n_x, \text{ then } |f_n(x) - f(x)| \leq a,$$

(i.e. $|f_n(x) - f(x)| \leq a$ eventually in n). If we suppose that

$$d = \sup\{d(f_n - f, C(X)) : n \in \mathbb{N}\} < +\infty,$$

then for each $\varepsilon > 0$, there exists $g \in \mathrm{conv}\{f_n : n \in \mathbb{N}\}$ such that

$$\|g - f\|_\infty < 2d + a + \varepsilon.$$

Proof. Without loss of generality, we may assume that $f = 0$. For each $x \in X$ define

$$F_x = \{n \in \mathbb{N} : |f_n(x)| \geq 2d + a + \varepsilon/2\}.$$

Since there exist n_x such that whenever $n > n_x$, $|f_n(x)| \leq a$, we obtain that F_x is finite. Put $\mathcal{F} = \{F_x : x \in X\}$. We show that $\mathcal{F}$ is well-founded. Suppose this is not the case. Then there is an infinite increasing sequence $(n_i)_i$ in $\mathbb{N}$ so that for each $k \in \mathbb{N}$, there exists an $x_k \in X$ such that $\{n_1, \ldots, n_k\} \subset F_{x_k}$. Since X is countably compact, $(x_k)_k$ must have a cluster point, say x_∞. Now fix $k \in \mathbb{N}$. Then for $j \geq k$, $n_k \in F_{x_j}$. Hence for all $j \geq k$, $|f_{n_k}(x_j)| \geq 2d + a + \varepsilon/2$. Since $d(f_{n_k}, C(X)) \leq d$, there exists a $g \in C(X)$ such that $d(f_{n_k}, g) < d + \varepsilon/8$. Hence for each $j \geq k$,

$$\begin{aligned} 2d + a + \varepsilon/2 \leq |f_{n_k}(x_j)| &\leq |f_{n_k}(x_j) - g(x_j)| + |g(x_j) - g(x_\infty)| \\ &\quad + |g(x_\infty) - f_{n_k}(x_\infty)| + |f_{n_k}(x_\infty)| \\ &\leq |f_{n_k}(x_\infty)| + 2d + \varepsilon/4 + |g(x_j) - g(x_\infty)|; \end{aligned}$$

thus

$$a + \varepsilon/4 \leq |f_{n_k}(x_\infty)| + |g(x_j) - g(x_\infty)|.$$

Since g is continuous and x_∞ is an accumulation point of $(x_j)_j$, $|g(x_j) - g(x_\infty)|$ can be made arbitrarily small. Therefore, $a + \varepsilon/4 \leq |f_{n_k}(x_\infty)|$ and this is true for each $k \in \mathbb{N}$, contradicting the assumption that $|f_n(x_\infty)| \leq a$ eventually as n goes to $+\infty$. We have proved that $\mathcal{F}$ is well-founded and then we may apply Pták Lemma to $\mathcal{F}$, so there is a $\mu \in \mathbf{M}$ such that $\mu(F_x) < \varepsilon/(3N)$ for each $x \in X$, where N is a uniform bound on (f_n). Choose $k \in \mathbb{N}$ such that $\operatorname{Supp}(\mu) \subset \{1, 2, \ldots, k\}$. Let $\lambda_i = \mu(i)$ for $1 \le i \le k$. Then $\sum_{i=1}^k \lambda_i = 1$. Let $x \in X$. Then for $i \notin F_x$, $|f_i(x)| < 2d + a + \varepsilon/2$. This yields:

$$\left|\sum_{i=1}^k \lambda_i f_i(x)\right| \le \sum_{i \in F_x \cap \{1,\ldots,k\}} |\lambda_i f_i(x)| + \sum_{i \in \{1,\ldots,k\} \setminus F_x} |\lambda_i f_i(x)|$$
$$\le N \sum_{i \in F_x \cap \{1,\ldots,k\}} \lambda_i + \sum_{i \in \{1,\ldots,k\} \setminus F_x} \lambda_i (2d + a + \frac{\varepsilon}{2})$$
$$\le N\mu(F_x) + 2d + a + \frac{\varepsilon}{2} \le N\frac{\varepsilon}{3N} + 2d + a + \frac{\varepsilon}{2} = 2d + a + \frac{5}{6}\varepsilon.$$

Hence $\| \sum_{i=1}^k \lambda_i f_i \|_\infty < 2d + a + \varepsilon$ as required. □

Corollary 4.2 (Mazur). *Suppose X is a countably compact space and $(f_n)_n$ is a uniformly bounded sequence of continuous functions defined on X pointwise convergent to a continuous function f. Then there exists a sequence $(g_n)_n$ in* $\operatorname{conv}\{f_n : n \in \mathbb{N}\}$ *that is $\| \cdot \|_\infty$-convergent to f.*

5. Distances to Baire one functions and oscillations of maps in two variables

For a function $f : X \times Y :\to \mathbb{R}$, let $f_x : Y \to \mathbb{R}$ and $f^y : X \to \mathbb{R}$ be the functions given by $f_x(y) = f(x, y) = f^y(x)$ for each $(x, y) \in X \times Y$. To the function f, we also associate the function $F : X \to \mathbb{R}^Y$ given by $F(x) = f_x$ for each $x \in X$. When we speak of oscillations of F, they are computed with respect to the uniform metric on $\mathbb{R}^Y$ as discussed in the Introduction. In the following, we recall the relationship between the oscillation of a function and its distance from the continuous ones. In the cited reference, the theorem is stated under more restricted conditions: X is paracompact and f is uniformly bounded on X. The proof in the reference has two parts, (i) and (ii). For the first part (i), one can find an outline of the proof for our case when X is normal in Engelking [7, Exercise 1.7.5 (b)]. The second part (ii) does not require f to be uniformly bounded.

Theorem 5.1 ([2, Proposition 1.18]). *Let X be a normal space. If $f \in \mathbb{R}^X$, then*

$$d(f, C(X)) = \frac{1}{2} \operatorname{osc}(f)$$

where

$$\operatorname{osc}(f) = \sup_{x \in X} \operatorname{osc}(f, x) = \sup_{x \in X} \inf\{\operatorname{diam} f(U) : U \subset X \text{ open}, x \in U\}.$$

The following is the main result of this section. Its proof relies on the terms and results given in Section 2. In addition, we need the following general facts which are easy to verify.

Fact (i) Let $\mathcal{U}$ be a discrete family of open subsets of a topological space X, and for each $U \in \mathcal{U}$, let C_U be a relatively closed subset of U. Then $D := \bigcup\{\overline{C_U} : U \in \mathcal{U}\}$ is closed in X and $\bigcup\{C_U : U \in \mathcal{U}\} = D \cap \bigcup \mathcal{U}$.

Fact (ii) Let $\mathcal{G}_\delta$ (resp. $\mathcal{F}_\sigma$) be the family of all G_δ (resp. F_σ) subsets of the space X. Then the intersection $\mathcal{G}_\delta \cap \mathcal{F}_\sigma$ is an algebra of subsets of X.

Theorem 5.2. *Let X be a metric space, K a compact space and $F : X \to \mathbb{R}^K$ a function. Then*

$$d(F, B_1(X, C(K))) \leq \frac{3}{2} \sup_{x \in X} \operatorname{osc}(F(x)) + 2 \sup_{k \in K} \operatorname{osc}(\pi_k \circ F), \tag{5.1}$$

where, for $F, F' : X \to \mathbb{R}^k$, $d(F, F') = \sup\{|F(x)(k) - F'(x)(k)| : (x, k) \in X \times K\}$ and the map $\pi_k : C(K) \to \mathbb{R}$ is given by $h \to h(k)$.

Proof. Obviously we may assume that the right hand side in (5.1) is finite.

Let $a > \sup_{x \in X} \operatorname{osc}(F(x))$ and $b > b' > \sup_{k \in K} \operatorname{osc}(\pi_k \circ F)$.

We first reduce the general case to the case where the image of the map F is uniformly bounded in $\mathbb{R}^X$.

By Theorem 5.1 for each k there exists $j^k \in C(X)$ such that $d(\pi_k \circ F, j^k) < b/2$. Define now the map $J : X \to \mathbb{R}^K$ by $J(x)(k) = j^k(x)$. Clearly J is continuous relative to τ_p and $d(F, J) \leq b/2$. For each $x \in X, \operatorname{osc} F(x)$ is finite that is $F(x)$ is a locally bounded function on K. Since K is compact, it follows that $F(x) \in \ell^\infty(X)$. Hence $J(x) \in \ell^\infty(K)$ also. Let $P_n = \{x \in X : \|J(x)\|_\infty \leq n\}$ and $Q_n = P_n \setminus P_{n-1}, P_0 = \emptyset$. Then each P_n is closed in X and $\{Q_n : n \in \mathbb{N}\}$ is a good partition of X by Remark 2.2(ii). Since $J(Q_n)$ is bounded in $\ell^\infty(K)$, $F(Q_n)$ is also bounded in $\ell^\infty(K)$ for each $n \in \mathbb{N}$. Suppose for each $n \in \mathbb{N}$ we can find a function H_n on Q_n which is constant on each set of a good partition of Q_n and $d(F|_{Q_n} - H_n) \leq 3a/2 + 2b$. Then let H be the function given by $H(x) = H_n(x)$ whenever $x \in Q_n$. By Remark 2.2, H is constant on each set of a good partition of X. Hence by Proposition 2.4, $H \in B_1(X, C(K))$ and $\|F - H\|_\infty \leq 3a/2 + 2b$. This finishes the proof. Therefore from now we assume that $F(X)$ is bounded in $\ell^\infty(K)$.

For each $x \in X$, by Theorem 5.1, there exists a function $g_x \in C(K)$ with $\|F(x) - g_x\|_\infty < a/2$. Define the map $G : X \to C(K)$ by $G(x) = g_x$. Then for each $(x, k) \in X \times K$, $|F(x)(k) - G(x)(k)| \leq a/2$.

Since X is metric, there is a σ-discrete base $\mathcal{U}$ for the topology of X, *i.e.* $\mathcal{U} = \bigcup\{\mathcal{U}_n : n \in \mathbb{N}\}$ where each $\mathcal{U}_n$ is a discrete family of non-empty open subsets of X. For each $U \in \mathcal{U}$, choose $x_U \in U$. For each n we define the function $G_n : \bigcup \mathcal{U}_n \to C(K)$ by $G_n(x) = G(x_U)$ for $x \in U \in \mathcal{U}_n$. Then the domain of G_n is $\bigcup \mathcal{U}_n$.

Now fix $x \in X$. Then for each $p \in \mathbb{N}$, there is an $n_p \in \mathbb{N}$ such that, for some $U_p \in \mathcal{U}_{n_p}, x \in U_p \subset B(x; 1/p)$. For each $k \in K$, $\operatorname{osc} \pi_k \circ F < b'$, and hence there is a $p_k \in \mathbb{N}$ such that $\operatorname{diam}(\pi_k \circ F(B(x; 1/p_k))) < b'$. If $p > p(k)$, since $U_p \subset B(x; 1/p)$,

$$\operatorname{diam}(\pi_k \circ F(U_p)) \leq \operatorname{diam} \pi_k \circ F(B(x; 1/p))) \leq \operatorname{diam}(\pi_k \circ F(B(x; 1/p_k))) < b'.$$

Furthermore because $x, x_{U_p} \in U_p$, we have

$$\begin{aligned} |G_{n_p}(x)(k) - F(x)(k)| &= |G(x_{U_p})(k) - F(x)(k)| \leq \\ &\leq |G(x_{U_p})(k) - F(x_{U_p})(k)| + |F(x_{U_p})(k) - F(x)(k)| < a/2 + b'. \end{aligned}$$

This shows that for each $k \in K, |G_{n_p}(x)(k) - F(x)(k)| < a/2 + b'$ eventually for p. Since $\operatorname{osc}(G_{n_p}(x) - F(x)) = \operatorname{osc} F(x) < a, d(G_{n_p}(x) - F(x), C(K)) < a/2$. It follows from Theorem 4.1 that there is a rational convex combination, say H, of $\{G_{n_p} : p \in \mathbb{N}\}$ such that $\|H(x) - F(x)\|_\infty < 3a/2 + b$. Let $\{H_m : m \in \mathbb{N}\}$ be an enumeration of all rational convex combinations of $\{G_n : n \in \mathbb{N}\}$. Then, as seen above, for each $x \in X$ there exists an $m \in \mathbb{N}$ such that $\|H_m(x) - F(x)\|_\infty \leq 3a/2 + b$. For $m \in \mathbb{N}$, let

$$A_m = \{x \in X : H_m(x) \text{ is defined and } \|H_m(x) - F(x)\|_\infty \leq 3a/2 + b\}.$$

Then $\bigcup\{A_m : m \in \mathbb{N}\} = X$. Let $\mathcal{V}_m$ be a discrete family of non-empty open sets such that $\bigcup \mathcal{V}_m = \mathrm{dom}(H_m)$ and such that H_m is constant on V for each $V \in \mathcal{V}_m$. Denote $h_m^V \in C(K)$ the constant value of H_m on V. Then

$$A_m = \bigcup\{V \cap \{x : \|F(x) - h_m^V\|_\infty \leq 3a/2 + b\} : V \in \mathcal{V}_m\}.$$

Put now

$$B_m = \bigcup\{V \cap \overline{\{x : \|F(x) - h_m^V\|_\infty \leq 3a/2 + b\}} : V \in \mathcal{V}_m\}.$$

Since $\mathcal{V}_m$ is discrete family of open sets in X, B_m is the intersection of a closed set and an open set in X by Fact (i) above. It follows that $B_m \in \mathcal{G}_\delta \cap \mathcal{F}_\sigma$ because X is metric.

Finally, define $L : X \to \ell^\infty(K)$ by

$$L(x) = H_m(x) \qquad \text{if } x \in C_m = B_m \setminus \bigcup_{k<m} B_k.$$

By Fact (ii) each C_m is a $\mathcal{F}_\sigma$ set. Hence $\{C_m : m \in \mathbb{N}\}$ is a good partition by Remark 2.2(ii). Note that relative to C_m, C_m is the union of the discrete family $\{V \cap C_m : V \in \mathcal{V}_m\}$ of open (hence F_σ) subsets and H_m is $C(K)$-valued and constant on each $V \cap C_m$. Hence by Remark 2.2(i), L is $C(K)$-valued and constant on each set of a good partition of X. Therefore by Proposition 2.4, $L \in B_1(X, C(K))$.

Now we have to prove that $\|L - F\|_\infty \leq 3a/2 + 2b$. For this, we only have to prove that if

$$x_0 \in \overline{\{x : \|F(x) - h_m^V\|_\infty \leq 3a/2 + b\}}, \tag{5.2}$$

then $\|F(x_0) - h_m^V\|_\infty \leq 3a/2 + 2b$.

Fix $k \in K$. Since $\mathrm{osc}(\pi_k \circ F) < b$, there exist an open neighborhood U_0 of x_0 such that $\mathrm{diam}(\pi_k \circ F(U_0)) < b$. By (5.2), there exist $x \in U_0$ such that $\|F(x) - h_m^V\|_\infty \leq 3a/2 + b$ so

$$|F(x_0)(k) - h_m^V(k)| \leq |F(x_0)(k) - F(x)(k)| + \\ + |F(x)(k) + h_m^V(k)| \leq b + 3a/2 + b = 3a/2 + 2b.$$

Since it is true for all $k \in K$, $\|F(x_0) - h_m^V\|_\infty \leq 3a/2 + 2b$. □

In particular, if E is a Banach space and $K = (B_{E^*}, w^*)$, we get the following result.

Corollary 5.3. *Let X be a metric space, E a Banach space and $F : X \to E^{**}$ a function, then*

$$d(F, B_1(X, E)) \leq (3/2) \sup_{x \in X} \mathrm{osc}(F(x)) + 2 \sup_{x^* \in B_{E^*}} \mathrm{osc}(x^* \circ F),$$

where $F(x)$ is considered a function on (B_{E^*}, w^*).

Proof. In the proof of the Theorem 5.2, if $K = (B_{E^*}, w^*)$, by [4, Corollary 4.2] we can choose $g_x \in E$, and so if we follow this proof, we get that

$$d(F, B_1(X, E)) \leq (3/2) \sup_{x \in X} \mathrm{osc}(F(x)) + 2 \sup_{x^* \in B_{E^*}} \mathrm{osc}(x^* \circ F).$$

□

Corollary 5.4. *Let X be a metric space, E a Banach space and $F : X \to E$ a function, then*

$$d(F, B_1(X, E)) \leq 2 \sup_{x^* \in B_{E^*}} \mathrm{osc}(x^* \circ F).$$

Theorem 5.2 and its Corollaries 5.3 and 5.4 are extensions of the following result.

Corollary 5.5 (Srivatsa, [20]). *Let X be a metric space, E a Banach space and K a compact space. Then:*

(i) *If $F : X \to (C(K), \tau_p)$, then F is Baire one.*
(ii) *If $F : X \to (E, weak)$ is continuous, then F is Baire one.*

Corollary 5.6. *Let X be a complete metric space, K a compact space and $f : X\times K \to \mathbb{R}$ a function. Then there exist a dense $\mathcal{G}_\delta$ set $D \subset X$ such that for each $x \in D$*

$$\operatorname{osc}(F, x) \leq 3 \sup_{x\in X} \operatorname{osc}(f_x) + 4 \sup_{k\in K} \operatorname{osc}(f^k)$$

where $F : X \to \mathbb{R}^K$ is the function defined by $F(x) = f_x$.

Proof. Put $a = (3/2) \sup_{x\in X} \operatorname{osc}(f_x)+2 \sup_{k\in K} \operatorname{osc}(f^k)$ and suppose that $a < +\infty$. By Theorem 5.2, $d(F, B_1(X, C(K))) \leq a$. Fix $\varepsilon > 0$, then there exists $G \in B_1(X, C(K))$ such that $\|F - G\|_\infty < a + \varepsilon/4$. Since X is a Baire space, there exist a dense $\mathcal{G}_\delta$ set $D_\varepsilon \subset X$ such that G is continuous at x for all $x \in D_\varepsilon$. Now we fix $x \in D_\varepsilon$. Then we can find a neighborhood U of x such that $\operatorname{diam} G(U) < \varepsilon/2$, and hence if $y, z \in U$,

$$\begin{aligned}\|F(y)-F(z)\|_\infty \leq& \\ &\leq \|F(y) - G(y)\|_\infty + \|G(y) - G(z)\|_\infty + \|G(z) - F(z)\|_\infty < \\ &< a + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + a + \frac{\varepsilon}{4} = 2a + \varepsilon.\end{aligned}$$

Therefore $\operatorname{osc}(F, x) \leq 2a + \varepsilon$. Now let $D = \bigcap\{D_{1/n} : n \in \mathbb{N}\}$. Then D is still a dense G_δ subset of X, and for each $x \in D$ $\operatorname{osc}(F, x) \leq 2a$. □

Lemma 3. *Let X be a topological space, K a compact space and $f : X \times K \to \mathbb{R}$ a function. Then for $x \in X$,*

$$\operatorname{osc}(F, x) \leq \sup_{k\in K} \operatorname{osc}(f, (x, k)) \leq 2 \operatorname{osc}(F, x) + \operatorname{osc}(f_x),$$

where $F : X \to \mathbb{R}^K$ is the function defined by $F(y) = f_y$ for each $y \in X$.

Proof. Let us prove the first inequality. Suppose that $a = \sup_{k\in K} \operatorname{osc}(f, (x, k))$ is finite and fix $\varepsilon > a$. Then for each $k \in K$, there exist an open neighborhood U_k of x and V_k of k such that $\operatorname{diam} f(U_k \times V_k) < \varepsilon$. Since $\{V_k : k \in K\}$ is an open cover of the compact space K, there exist $k_1, k_2, \ldots, k_n$ such that the family $\{V_{k_i} : 1 \leq i \leq n\}$ covers K. Put $U = \bigcap_{i=1}^n U_{k_i}$. Then U is an open neighborhood of x. Choose $x', x'' \in U$ and $k \in K$. Then there exists an $i \in \{1, \ldots, n\}$ such that $k \in V_{k_i}$. Since $(x', k), (x'', k) \in U_{k_i} \times V_{k_i}$ and $\operatorname{diam} f(U_{k_i} \times V_{k_i}) < \varepsilon$ then

$$|f(x', k) - f(x'', k)| < \varepsilon$$

and since we can do it for all $k \in K$ then

$$d(F(x'), F(x'')) \leq \varepsilon$$

so $\operatorname{osc}(F, x) \leq \varepsilon$ and since we can do it with each $\varepsilon > a$ the first inequality is proved.

For the second inequality, suppose that $\operatorname{osc}(F, x)$ and $\operatorname{osc}(f_x)$ are finite and choose $\varepsilon > \operatorname{osc}(F, x)$ and $\delta > \operatorname{osc}(f_x)$. Fix $k \in K$, we have to prove that $\operatorname{osc}(f, (x, k)) \leq 2\varepsilon+\delta$. Since $\operatorname{osc}(F, x) < \varepsilon$ there exist a neighborhood U of x such that $d(f_x, f_y) < \varepsilon$ for each $y \in U$. Since $\operatorname{osc}(f_x) < \delta$, there exist a neighborhood V of k such that $\operatorname{diam} f_x(V) < \delta$.

Then, for $(x', k'), (x'', k'') \in U \times V$

$$\begin{aligned} |f(x',k') - f(x'',k'')| &\leq |f(x',k') - f(x,k')| + |f(x,k') - f(x,k'')| + \\ &+ |f(x,k'') - f(x'',k'')| < \\ &< \varepsilon + \delta + \varepsilon = 2\varepsilon + \delta \end{aligned}$$

and then $\mathrm{osc}(f,(x,k)) \leq 2\varepsilon + \delta$. □

Corollary 5.7. *Let X be a complete metric space, K a compact space and $f : X \times K \to \mathbb{R}$ a function. Then there exists a dense $\mathcal{G}_\delta$ set $D \subset X$ such that for each $(y,k) \in D \times K$*

$$\mathrm{osc}(f,(y,k)) \leq 7 \sup_{x \in X} \mathrm{osc}(f_x) + 8 \sup_{k \in K} \mathrm{osc}(f^k).$$

Proof. Apply Corollary 5.6 and Lemma 3. □

Corollary 5.8 ([17])**.** *Let X be a complete metric space, K a compact space and $f : X \times K \to \mathbb{R}$ a separately continuous function. Then there exist a dense $\mathcal{G}_\delta$ set $D \subset X$ such that f is continuous in each $(x,k) \in D \times K$.*

6. Games and Oscillation of Maps in Two Variables

Corollary 5.8 of the previous section is an answer to a more general question of the following type: For what Baire space X and compact space K, does the following statement $N(X,K)$ hold?

> $N(X,K)$: If $f : X \times K \to \mathbb{R}$ is separately continuous, *i .e.* f_x and f^k are continuous for each $(x,k) \in X \times K$, then for some dense G_δ subset D of X, the function f is continuous (relative to the product topology) at each point $(x,k) \in D \times K$.

A Baire space X is said to have property $\mathcal{N}$ if $N(X,K)$ holds for each compact Hausdorff space K, and similarly a compact space K is said to have property $\mathcal{N}^*$ if $N(X,K)$ holds for each Baire space X. Corollary 5.8 says that complete metric spaces have property $\mathcal{N}$, and Corollary 5.7 is a quantitative version of $\mathcal{N}$. We know, for instance, countable Čech-complete (= strongly countably complete) spaces (defined below) have the property $\mathcal{N}$ [17] and Valdivia-compact spaces have the property$\mathcal{N}^*$ [6]. In this section we examine quantitative versions of properties $\mathcal{N}$ and $\mathcal{N}^*$ for even larger classes of spaces with $\mathcal{N}$ or $\mathcal{N}^*$. These spaces are defined in terms of topological games. Therefore we begin the section with the topological games. All games are infinite games for two players, α and β, and each moves alternatively. The *board* is a topological space X. The Banach-Mazur game $\mathcal{G}(X)$ in X is played as follows: First β chooses a non-empty open set U_0 in X (β's 0-th *move*). Then α chooses a non-empty open subset V_0 of U_0 (α's 0-th move). Then the first move by β is a non-empty open subset U_1 of V_0 followed by α's first move $V_1 \subset U_1$, and so on. Inductively the player β's n-th move is a non-empty open set $U_n \subset V_{n-1}$ followed by α's n-th move: non-empty open set $V_n \subset U_n$. The player α is said to win the game if $\bigcap\{V_n : n \in \mathbb{N}\} = \bigcap\{U_n : n \in \mathbb{N}\} \neq \emptyset$. Otherwise β wins the game.

A *strategy* s for α in the game $\mathcal{G}(X)$ is a rule which determines α's move at each stage based on the game played so far following the strategy. Thus at the n-th stage, α's move V_n is given as $V_n = s(U_0, U_1, ..., U_n)$, where $U_{i+1} \subset s(U_0, U_1, ..., U_i)$ for $i \in \{0, 1, ..., n-1\}$. The strategy s is called a *winning strategy* if α wins whenever it uses the strategy s. A strategy for β is similarly defined by switching the sides. If α has a winning strategy in the game $\mathcal{G}(X)$, then the space X is said to be *α-favorable*. If β does not have a winning strategy, then X is said to be *β-unfavorable*. Krom in [16] and Saint

Raymond in [19] have shown independently that *a topological space X is a Baire space if and only if X is β-unfavorable*.

We now consider a modification $\mathcal{G}_\sigma(X)$, due to Christensen [5], of the Banach-Mazur game. As before the players α and β move alternatively beginning with β. The moves for β are the same as before: at the n-th stage, β's move is a non-empty open $U_n \subset V_{n-1}$. Then α's n-th move is a pair (V_n, x_n) where V_n is a non-empty open subset of U_n and $x_n \in X$. The player α wins the game $\mathcal{G}_\sigma(X)$ if $\bigcap\{V_n : n \in \mathbb{N}\} = \bigcap\{U_n : n \in \mathbb{N}\}$ contains a cluster point of the sequence $(x_n)_n$. A strategy in the game $\mathcal{G}_\sigma(X)$ is defined analogously as in the game $\mathcal{G}(X)$. A space X is said to be *σ-β-unfavorable* if the player β does not have a winning strategy in $\mathcal{G}_\sigma(X)$. From the result cited in the last paragraph, it is clear that σ-β-unfavorable spaces are Baire spaces. Generalizing a Christensen's result, Saint Raymond [19] proved the following theorem in the case $\sup_{x\in x} \operatorname{osc}(f_x) = 0$ and $\sup_{k\in K} \operatorname{osc}(f^k) = 0$. Later Bouziad [3] gave it a short and elegant proof. Our proof below uses Bouziad's ideas.

Theorem 6.1. *Let $f : X \times K \to \mathbb{R}$ be a map, where X is a σ-β-unfavorable space and K is a compact space. Then there exists a dense G_δ-subset D of X such that, for each $(y, k) \in D \times K$,*

$$\operatorname{osc}(f, (y, k)) \le 6 \sup_{x\in X} \operatorname{osc}(f_x) + 8 \sup_{k\in K} \operatorname{osc}(f^k).$$

Proof. Let $b = \sup_{k\in K} \operatorname{osc}(f^k)$, $c = \sup_{x\in X} \operatorname{osc}(f_x)$ and let $r = 6c + 8b$. We may assume that r is finite, otherwise the assertion is trivially true. For $n \in \mathbb{N}$, let

$$A_n = \{x \in X : \operatorname{osc}(f, (x, k)) < r + 1/n \text{ for each } k \in K\}.$$

Since oscillation is upper semicontinuous and K compact, A_n is open.

We show each A_n is dense in X by contradiction. So suppose for some $p \in \mathbb{N}$, $\overline{A_p} \ne X$. We now define a strategy s for β for the game $\mathcal{G}_\sigma(X)$. First let $U_0 := s(\emptyset) = X \setminus \overline{A_p}$. Inductively, suppose $(V_0, a_0), (V_1, a_1), ..., (V_{n-1}, a_{n-1})$ have been played by α. We must define β's response

$$U_n := s((V_0, a_0), (V_1, a_1), ..., (V_{n-1}, a_{n-1})).$$

First choose an $x_n \in V_{n-1}$. Since $x_n \notin A_p$, there exists a $k_n \in K$ such that

$$\operatorname{osc}(f, (x_n, k_n)) \ge r + 1/p. \tag{6.1}$$

Choose an open subset O_n of X such that $x_n \in O_n \subset V_{n-1}$ and

$$\operatorname{diam} f(O_n, k_n) < b + 1/n. \tag{6.2}$$

For each $x \in X$, since $\operatorname{osc} f_x \le c$, Theorem 5.1 allows us to pick $g_x \in C(K)$ such that $\|f_x - g_x\|_\infty < c/2 + 1/(36p)$. Now define the map $g : X \times K \to \mathbb{R}$ by $g(x, k) = g_x(k)$. Clearly

$$|f(x, k) - g(x, k)| < c/2 + 1/(36p) \quad \text{for} \quad (x, k) \in X \times K. \tag{6.3}$$

Next we let

$$W_n = \{k \in K : |g(a_i, k_n) - g(a_i, k)| < 1/n \quad \text{for} \quad i = 0, 1, ..., n-1\}. \tag{6.4}$$

Since g_{a_i} is continuous for $i = 0, 1, ..., n-1$, W_n is an open neighborhood of k_n in K. By (6.1), there is a point $(x'_n, k'_n) \in O_n \times W_n$ such that

$$|f(x_n, k_n) - f(x'_n, k'_n)| > r/2 + 1/(3p). \tag{6.5}$$

For, otherwise $\operatorname{osc}(f,(x_n,k_n)) \leq r + 2/(3p)$. Choose an open set U_n in X such that $x'_n \in U_n \subset O_n$ and

$$\operatorname{diam} f(U_n,k'_n) < b + 1/n. \tag{6.6}$$

This U_n is the n-th move of β. Since the space X is σ-β-unfavorable, the strategy s for β above is not a winning one. Thus there is a winning play for α against β's strategy s. Let $(V_0,a_0),(V_1,a_1),...,(V_n,a_n),...$ be such a play for α. Then the sequence $(a_n)_n$ has a cluster point $a \in \bigcap_{n=0}^{\infty} U_n$. Also let (k_∞,k'_∞) be a cluster point of the sequence $(k_n,k'_n)_n$.

Since $k'_n \in W_n$, by (6.4) we have

$$|g(a_i,k_n) - g(a_i,k'_n)| < 1/n \ \text{ for } \ 0 \leq i < n$$

and then, since g_{a_i} is continuous

$$g(a_i,k_\infty) = g(a_i,k'_\infty) \quad \text{for each} \quad i \in \mathbb{N}. \tag{6.7}$$

By (6.5)

$$\begin{aligned} r/2 + 1/(3p) &< |f(x_n,k_n) - f(x'_n,k'_n)| \leq \\ &\leq |f(x_n,k_n) - f(a,k_n)| + |f(a,k_n) - f(a,k'_n)| + \\ &+ |f(a,k'_n) - f(x'_n,k'_n)|. \end{aligned}$$

Since $x_n, a \in O_n$ and $x'_n, a \in U_n$, by (6.2) and (6.6) the first and the third terms of the last member of the last inequality are less than $b + 1/n$. Consequently we have

$$\begin{aligned} r/2 + 1/(3p) &< |f(a,k_n) - f(a,k'_n)| + 2b + 2/n < \\ &< |g(a,k_n) - g(a,k'_n)| + c + 1/(18p) + 2b + 2/n \end{aligned}$$

for each n. Here, in the last inequality, we have used (6.3). Since g_a is continuous, it follows that

$$r/2 + 1/(3p) \leq c + 1/(18p) + 2b + |g(a,k_\infty) - g(a,k'_\infty)|. \tag{6.8}$$

By our definition of b, there is an open neighborhood G of a in X such that

$$\operatorname{diam} f(G,k_\infty) < b + 1/(12p) \text{ and } \operatorname{diam} f(G,k'_\infty) < b + 1/(12p).$$

Since a is a cluster point of the sequence $(a_n)_n$, $a_i \in G$ for some i. Therefore using (6.7) and (6.3)

$$\begin{aligned} &|g(a,k_\infty) - g(a,k'_\infty)| \leq \\ &\quad \leq |g(a,k_\infty) - g(a_i,k_\infty)| + |g(a_i,k'_\infty) - g(a,k'_\infty)| \leq \\ &\quad \leq |f(a,k_\infty) - f(a_i,k_\infty)| + |f(a_i,k'_\infty) - f(a,k'_\infty)| + 2c + 1/(9p) < \\ &\quad < 2c + 2b + 5/(18p). \end{aligned}$$

Combining this inequality with (6.8), we obtain that

$$r/2 + 1/(3p) < 3c + 4b + 1/(3p),$$

from which we conclude that $r < 6c + 8b$, contradicting the definition of r. This proves that A_n is dense for each n. Let $D = \bigcap\{A_n : n \in \mathbb{N}\}$. Then since σ-β-unfavorable spaces are Baire, D is a dense G_δ subset of X. By the definition of A_n, it is clear that D has the stated property. □

Corollary 6.2. *Let X and K be as in the theorem and let $F : X \to C(K)$ be a map. Then there exists a dense G_δ-subset D of X such that, for each $x \in D$,*

$$\operatorname{osc}(F, x) \le 8 \sup_{k\in K} \operatorname{osc}(\pi_k \circ F),$$

where the map $\pi_k : C(K) \to \mathbb{R}$ is given by $h \mapsto h(k)$.

Proof. Apply the theorem to the map $f : X \times K \to \mathbb{R}$ defined by $f(x, k) = F(x)(k)$ to obtain a dense G_δ-subset D of X. The conclusion follows from the inequality: $\operatorname{osc}(F, x) \le \sup_{k\in K} \operatorname{osc}(f, (x, k))$, see Lemma 3. □

Corollary 6.3. *Let X be a normal σ-β-unfavorable space, let K be a compact space and let $f : X \times K \to \mathbb{R}$ be a map. Then there exists a dense G_δ-subset D of X such that, for each $(y, k) \in D \times K$,*

$$\operatorname{osc}(f, (y, k)) \le 6 \sup_{x\in X} \operatorname{osc}(f_x) + 7 \sup_{k\in K} \operatorname{osc}(f^k).$$

Proof. Let $c = \sup_{x\in X} \operatorname{osc}(f_x)$ and $b = \sup_{k\in K} \operatorname{osc}(f^k)$ and $r > b$. We may assume that b is finite, otherwise the assertion is trivially true. By Theorem 5.1, for each $k \in K$, there is a $g^k \in C(X)$ such that $\|f^k - g^k\|_\infty < r/2$. Now define the map $g : X \times K \to \mathbb{R}$ by $g(x, k) = g^k(x)$. Clearly $|f(x, k) - g(x, k)| < r/2$ for $(x, k) \in X \times K$. This makes oscillations of f, f_x, f^k and g, g_x, g^k differ by at most r respectively. Applying Theorem 6.1 to the map g we get that there exists a G_δ-subset D of X such that, for each $(x, k) \in D\times K,\ \operatorname{osc}(g, (x, y)) \le 6 \sup_{x\in X} \operatorname{osc}(g_x)+8 \sup_{k\in K} \operatorname{osc}(g^k) \le 6(c+r)+8\cdot 0$. It follows for such (x, k),

$$\operatorname{osc}(f, (x, k)) \le \operatorname{osc}(g, (x, k)) + r \le 6(c + r) + r = 6c + 7r.$$

The assertion now follows from this. □

Remark 6.4. Notice that the assumptions of the last corollary and Theorem 6.1 are very similar. The only difference is that the former assumes the normality of X. This extra assumption has the effect of improving the conclusion slightly. As we show below, complete metric spaces satisfy the assumptions of Corollary 6.3. So it is a generalization of our Corollary 5.7 with better estimates. This is the result of our more direct approach of this section.

Here are examples of σ-β-unfavorable spaces. Separable Baire spaces are σ-β-unfavorable spaces (Saint-Raymond [19]). A completely regular space X is called *countably Čech-complete* (or *strongly countably complete*) if there is a sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open coverings of X such that $\bigcap\{F_n : n \in \mathbb{N}\} \ne \emptyset$ whenever $\{F_n : n \in \mathbb{N}\}$ is a decreasing sequence of closed subsets of X such that F_n is contained in some member of $\mathcal{U}_n$ for each n. Christensen [5] has shown that countably Čech-complete are σ-β-unfavorable. In particular, Čech-complete spaces are σ-β-unfavorable. So locally compact spaces and complete metric spaces are all σ-β-unfavorable.

In order to state the next theorem, we need one more game. Let Y be a topological space and let L be a dense subset of $Y \times Y$. The game $\mathcal{G}(\Delta, L, Y)$ goes as follows. First the player α chooses an open neighborhood W_0 of the diagonal Δ in $Y\times Y$ and the player β chooses a point $(a_0, b_0) \in W_0 \cap L$. At the n-th stage α chooses an open neighborhood W_n of Δ and β chooses a point $(a_n, b_n) \in W_n \cap L$. The player α is said to win the game if, for each neighborhood W of Δ, $(a_n, b_n) \in W$ for infinitely many n's. For the case $L = Y\times Y$, this game is defined and very effectively used by Bouziad [3]. It is a variation of the game defined earlier by Gruenhage [13]. If α has a winning strategy in the game $\mathcal{G}(\Delta, L, Y)$, then Y is called a $\mathcal{G}(\Delta, L)$-α-*favorable space*.

Theorem 6.5. *Let X be a Baire space and let K be a compact $\mathcal{G}(\Delta, L)$-α-favorable space for some dense subset L of $K \times K$. Suppose $f : X \times K \to \mathbb{R}$ is a map such that $f_x \in C(K)$ for each $x \in X$. Then there exists a dense G_δ subset D of X such that for each $(x, k) \in D \times K$,*

$$\operatorname{osc}(f, (x, k)) \leq 6 \sup_{k \in K} \operatorname{osc}(f^k).$$

Proof of the Theorem. We start by isolating the following very technical part of the proof as a lemma for further comments and use.

Lemma 4. *Under the assumptions of Theorem 6.5, if we let $b = \sup_{k \in K} \operatorname{osc}(f^k)$ and if for some $r > 0$ and $p \in \mathbb{N}$,*

$$A = \{x \in X : \operatorname{osc}(f, (x, k)) < r + \frac{1}{p} \quad \textit{for each} \quad k \in K\}$$

is not dense, then there exists a sequence $(k_n, k_n')_n$ in L and $a \in X$ such that for each open neighborhood W of Δ, $(k_n, k_n') \in W$ for infinitely many n's and

$$r/2 + 1/(3p) < 3b + 3/n + |f(a, k_n) - f(a, k_n')| \quad \textit{for all} \quad n \in \mathbb{N}. \tag{6.9}$$

Proof of the Lemma. Since oscillation is upper semicontinuous and K compact, A is open. Suppose that $\overline{A} \neq X$. Assume now that α and β play games $\mathcal{G}(X)$ and $\mathcal{G}(\Delta, L, K)$ simultaneously. Let s be the winning strategy for α in $\mathcal{G}(\Delta, L, K)$, we are going to define a strategy t for β in $\mathcal{G}(X)$ step by step.

So let $U_0 = t(\emptyset) := X \setminus \overline{A}$ and let $(k_0, k_0') \in W_0 \cap L$ be the 0-th move by β in the game $\mathcal{G}(\Delta, L, K)$, where $W_0 = s(\emptyset)$. Assume that we are at the $(n-1)$-th stage. In the game $\mathcal{G}(\Delta, L, K)$, β's moves so far are $(k_0, k_0'), (k_1, k_1'), ..., (k_{n-1}, k_{n-1}')$. In the game $\mathcal{G}(X)$, α's moves so far are $V_0, V_1, ..., V_{n-1}$. We must define β's n-th move $U_n = t(V_0, V_1, ..., V_{n-1})$.

First choose $x_n \in V_{n-1}$. Since $x_n \notin A$, there exists $y_n \in K$ such that

$$\operatorname{osc}(f, (x_n, y_n)) \geq r + 1/p. \tag{6.10}$$

By definition of b, there is an open set O_n in X such that $x_n \in O_n \subset V_{n-1}$ and

$$\operatorname{diam} f(O_n, y_n) < b + 1/n. \tag{6.11}$$

Now $W_n := s((k_0, k_0'), (k_1, k_1'), ..., (k_{n-1}, k_{n-1}'))$ is an open neighborhood of Δ. Define

$$G_n = \{y \in K : (y_n, y) \in W_n\}. \tag{6.12}$$

Then G_n is an open neighborhood of y_n in K. By (6.10), there exists $(x_n', y_n') \in O_n \times G_n$ such that

$$|f(x_n, y_n) - f(x_n', y_n')| > r/2 + 1/(3p). \tag{6.13}$$

Since $x_n, x_n' \in O_n$, we have by (6.11)

$$|f(x_n, y_n) - f(x_n', y_n)| < b + 1/n. \tag{6.14}$$

Since $y_n' \in G_n$, by (6.12)

$$(y_n, y_n') \in W_n. \tag{6.15}$$

Since $W_n \cap L$ is dense in W_n and f is continuous in the second variable with the first variable fixed, (6.13) and (6.14) are still valid when (y_n, y_n') is replaced by some $(k_n, k_n') \in W_n \cap L$. Thus we have

$$|f(x_n, k_n) - f(x_n', k_n')| > r/2 + 1/(3p) \tag{6.16}$$

and

$$|f(x_n,k_n)-f(x'_n,k_n)|<b+1/n. \tag{6.17}$$

We let (k_n,k'_n) be β's n-th move.

Finally, there exists an open set U_n in X such that $x'_n\in U_n\subset O_n$ and

$$\operatorname{diam} f(U_n,k_n)<b+1/n \qquad \text{and} \qquad \operatorname{diam} f(U_n,k'_n)<b+1/n. \tag{6.18}$$

We then let $U_n=t(V_0,V_1,...,V_{n-1})$. This completes the definition of t. Since X is a Baire space, t is not a winning strategy for β. Let $V_0,V_1,...,V_n,...$ be a winning play for α against β's strategy t. Then there exists an element $a\in\bigcap_{n=0}^{\infty}V_n=\bigcap_{n=0}^{\infty}U_n$. Hence by (6.16),

$$\begin{aligned}r/2+1/(3p)&<|f(x_n,k_n)-f(x'_n,k'_n)|\le|f(x_n,k_n)-f(x'_n,k_n)|\\&\quad+|f(x'_n,k_n)-f(a,k_n)|+|f(a,k_n)-f(a,k'_n)|+|f(a,k'_n)-f(x'_n,k'_n)|.\end{aligned}$$

Using (6.17) and (6.18) twice, we have for each $n\in\mathbb{N}$,

$$r/2+1/(3p)<3b+3/n+|f(a,k_n)-f(a,k'_n)|.$$

Since s is a winning strategy for the game $\mathcal{G}(\Delta,L,K)$, $(k_n,k'_n)\in W$ for infinitely many n's. □

Proof of the Theorem 6.5 resumed: Let $b=\sup_{k\in K}\operatorname{osc}(f^k)$ and $r=6b$. For each $n\in\mathbb{N}$, let

$$A_n=\{x\in X:\operatorname{osc}(f,(x,k))<r+1/n \text{ for each } k\in K\}.$$

We show by contradiction that A_n is dense in X for each $n\in\mathbb{N}$. So assume that for some p, $\overline{A_p}\neq X$. Then, by Lemma 4, there exists a sequence $(k_n,k'_n)_n$ in L and $a\in X$ such that for each neighborhood W of Δ, $(k_n,k'_n)\in W$ for infinitely many n's and

$$r/2+1/(3p)<3b+3/n+|f(a,k_n)-f(a,k'_n)| \quad \text{for each} \quad n\in\mathbb{N}.$$

Let

$$W=\{(k,k')\in K\times K:|f(a,k)-f(a,k')|<1/(12p)\}.$$

W is open so we can pick an $n>18p$ with $(k_n,k'_n)\in W$. Then $r/2+1/(3p)<3b+1/(6p)+1/(12p)$ which implies $r<6b$ contrary to the definition of r. This proves that A_n is dense in X for each $n\in\mathbb{N}$. Since X is a Baire, $D:=\bigcap_{n=1}^{\infty}A_n$ satisfies the conditions of the theorem. □

Remark 6.6. In Lemma 4, if $L=K\times K$, then the proof is a bit simpler, the condition $f_x\in C(K)$ for each $x\in X$ is not needed and the inequality (6.9) can be sharpened. Indeed, under the hypothesis: $L=K\times K$, the sequence (y_n,y'_n) can be taken to be (k_n,k'_n). By so doing, inequality (6.9) can be replaced by

$$r/2+1/(3p)<2b+2/n+|f(a,k_n)-f(a,k'_n)|.$$

The following corollary is proved using Theorem 6.5 in the same way as Corollary 6.2 is derived from Theorem 6.1.

Corollary 6.7. *Let X be a Baire space, let K be a $\mathcal{G}(\Delta,L)$-α-favorable compact space for some dense subset L of $K\times K$, and let $F:X\to C(K)$ be a map. Then there exists a dense G_δ-subset D of X such that, for each $x\in D$, $\operatorname{osc}(F,x)\le 6\sup_{k\in K}\operatorname{osc}(\pi_k\circ F)$.*

Corollary 6.8. *Let X be a Baire space, let K be a $\mathcal{G}(\Delta, L)$-α-favorable compact space for some dense subset L of $K \times K$, and let $f : X \times K \to \mathbb{R}$ be a map. Then there exists a dense G_δ-subset D of X such that, for each $(y,k) \in D \times K$ we have*

$$\operatorname{osc}(f,(y,k)) \leq 7 \sup_{x \in X} \operatorname{osc}(f_x) + 6 \sup_{k \in K} \operatorname{osc}(f^k).$$

Proof. Let $c = \sup_{x\in X} \operatorname{osc}(f_x)$ and $b = \sup_{k\in K} \operatorname{osc}(f^k)$. As usual we may assume that c and b are finite. Let $r > c$. Then for each $x \in X$, there is a $g_x \in C(K)$ such that $\|f_x - g_x\|_\infty < r/2$ by Theorem 5.1. Define $g : X \times K \to \mathbb{R}$ by $g(x,k) = g_x(k)$. Then $|f(x,y) - g(x,k)| < r/2$ for each $(x,k) \in X \times K$. As remarked in the proof of Corollary 6.3, the oscillations of f, f_x, f^k and g, g_x, g^k respectively differ by at most r. By Theorem 6.5, there is a dense $\mathcal{G}_\delta$ subset D of X such that $osc(g,(x,k)) \leq 6 \sup_{k\in K} \operatorname{osc}(g^k) \leq 6(b+r)$ for each $(x,k) \in D \times K$. Therefore for each $(x,k) \in D \times K$,

$$\operatorname{osc}(f,(x,k)) \leq \operatorname{osc}(g,(x,k)) + r \leq 6(b+r) + r = 7r + 6b.$$

The corollary follows from this. □

The next two results are for the special case when $L = K \times K$. They apply, for instance, to the case when K is Corson compact (see below). Notice that the results are sharper than for more general cases above.

Theorem 6.9. *Let X be a Baire space and let K be a compact $\mathcal{G}(\Delta, K \times K)$-$\alpha$-favorable space. Let $f : X \times K \to \mathbb{R}$ be a map. Then there exists a dense G_δ subset D of X such that for each $(y,k) \in D \times K$,*

$$\operatorname{osc}(f,(y,k)) \leq 2 \sup_{x \in X} \operatorname{osc}(f_x) + 4 \sup_{k \in K} \operatorname{osc}(f^k).$$

Proof. Let $c = \sup_{x\in X} \operatorname{osc}(f_x)$, $b = \sup_{k\in K} \operatorname{osc}(f^k)$ and $r = 2c + 4b$. For each $n \in \mathbb{N}$, let

$$A_n = \{x \in X : \operatorname{osc}(f,(x,k)) < r + 1/n \ \text{ for each } k \in K\}.$$

We show by contradiction that A_n is dense in X for each $n \in \mathbb{N}$. So assume that for some p, $\overline{A_p} \neq X$. Then, by Remark 6.6 there exist a sequence $(k_n, k'_n)_n$ in $K \times K$ and $a \in X$ such that for each neighborhood W of Δ, $(k_n, k'_n) \in W$ for infinitely many n's and

$$r/2 + 1/(3p) < 2b + 2/n + |f(a,k_n) - f(a,k'_n)| \quad \text{for each} \quad n \in \mathbb{N}. \tag{6.19}$$

For each $x \in X$, $\operatorname{osc}(f_x) \leq c$ so by Theorem 5.1 there exist $g_x \in C(X)$ such that $\|f_x - g_x\|_\infty < c/2 + (1/18p)$. Now define the map $g : X \times K \to \mathbb{R}$ by $g(x,k) = g_x(k)$. Clearly

$$|f(x,k) - g(x,k)| < c/2 + 1/(18p) \quad \text{for} \quad (x,k) \in X \times K. \tag{6.20}$$

In particular,

$$|f(a,k_n) - f(a,k'_n)| \leq c + 1/(9p) + |g(a,k_n) - g(a,k'_n)|. \tag{6.21}$$

Define

$$W = \{(k,k') \in K \times K : |g(a,k) - g(a,k')| < 1/(9p)\}.$$

W is open so we can pick an $n > 18p$ with $(k_n, k'_n) \in W$. Then by (6.19) and (6.21)

$$r/2 + 1/(3p) < 2b + 1/(9p) + c + 2/(18p) + |g(a,k_n) - g(a,k'_n)| < c + 2b + 1/(3p)$$

which implies $r < 2c + 4b$ contrary to the definition of r. This proves that A_n is dense in X for each $n \in \mathbb{N}$. Since X is a Baire, $D := \bigcap_{n=1}^\infty A_n$ satisfies the conditions of the theorem. □

The next corollary follows from the previous one in exactly same manner as Corollary 6.3 does from Theorem 6.1

Corollary 6.10. *Let X be a normal Baire space and let K be a compact $\mathcal{G}(\Delta, K \times K)$-$\alpha$-favorable space. Let $f : X \times K \to \mathbb{R}$ be a map. Then there exists a dense G_δ subset D of X such that for each $(y, k) \in D \times K$,*

$$\operatorname{osc}(f, (y, k)) \leq 2 \sup_{x \in X} \operatorname{osc}(f_x) + 3 \sup_{k \in K} \operatorname{osc}(f^k).$$

Let Γ be an arbitrary set and let

$$\Sigma(\Gamma) := \{x \in [0, 1]^\Gamma : \{\gamma \in \Gamma : x(\gamma) \neq 0\} \text{ is countable}\}.$$

A compact space K is said to be a *Corson compact* space if it is homeomorphic to a subset of $\Sigma(\Gamma)$, where $\Sigma(\Gamma)$ is given the relativization of the product topology on $[0, 1]^\Gamma$. The space K is called a *Valdivia compact* space if it is homeomorphic to a compact subset K' of $[0, 1]^\Gamma$ in such a way that $K' \cap \Sigma(\Gamma)$ is dense in K'.

Bouziad [3] has shown that Corson compact spaces K are $\mathcal{G}(\Delta, K \times K)$-$\alpha$-favorable. By modifying his proof a little, one can show each Valdivia compact space K is $\mathcal{G}(\Delta, L)$-α-favorable for a suitable dense subset L of $K \times K$. For completeness, we present the proof.

Lemma 5. *Let K be a compact space, and let L be a dense subset of $K \times K$. Assume that there is an open covering $\mathcal{U}$ of $K \times K \setminus \Delta$ such that*

(i) *the closure of each member of $\mathcal{U}$ is disjoint from Δ,*
(ii) *each point of L is contained in at most countable number of members of $\mathcal{U}$.*

Then K is $\mathcal{G}(\Delta, L)$-α-favorable.

Proof. (Bouziad) For each $(x, y) \in L$, let $\{U_n(x, y) : n \in \mathbb{N}\}$ be an enumeration of all members of $\mathcal{U}$ containing (x, y). We allow (possibly) infinite repetitions and set $U_n(x, y) = \emptyset$ if $(x, y) \in \Delta \cap L$. A strategy s for α in the game $\mathcal{G}(\Delta, L, K)$ is defined as follows: Let $s(\emptyset) = K \times K$ and at the n-th stage, assuming β's move so far has been $(x_0, y_0), (x_1, y_1), ..., (x_{n-1}, y_{n-1})$ in L, we let

$$t((x_0, y_0), (x_1, y_1), ..., (x_{n-1}, y_{n-1})) = (K \times K) \setminus \cup\{\overline{U_i(x_j, y_j)} : 0 \leq i, j \leq n - 1\}.$$

We show that this is a wining strategy. Fix (x, y) is a cluster point of the sequence $(x_n, y_n)_n$. Suppose $(x, y) \notin \Delta$. Then $(x, y) \in U$ for some $U \in \mathcal{U}$. It follows $(x_k, y_k) \in U$ for some k and consequently, for some p, $U = U_p(x_k, y_k)$. Then for all $n > \max(k, p)$, $(x_n, y_n) \notin U$. This contradicts that (x, y) is a cluster point of $(x_n, y_n)_n$. So $(x, y) \in \Delta$. Hence if W is a neighborhood of Δ, then $(x_n, y_n) \in W$ for infinitely many n's. □

Corollary 6.11. *Each Valdivia-compact space K is $\mathcal{G}(\Delta, D \times D)$-$\alpha$-favorable for some dense subset D of K.*

Proof. (Bouziad) Assume that $K \subset [0, 1]^\Gamma$ and that $D = K \cap \Sigma(\Gamma)$ is dense in K. For each $(\gamma, n) \in (\Gamma, \mathbb{N})$, let $U_{(\gamma,n)} = \{(x, y) \in K \times K : |x(\gamma) - y(\gamma)| > 1/n\}$. Then it is easy to check that $\mathcal{U} = \{U_{(\gamma,n)} : (\gamma, n) \in (\Gamma, \mathbb{N})\}$ satisfies conditions (i),(ii) of the Lemma 5. □

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DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DE MURCIA, 30.100 ESPINARDO MURCIA, SPAIN

E-mail address: angosto@um.es

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DE MURCIA, 30.100 ESPINARDO MURCIA, SPAIN

E-mail address: beca@um.es

DEPARTMENT OF MATHEMATICS, BOX 354350, UNIVERSITY OF WASHINGTON, SEATTLE, WASHINGTON 98195–4350, U.S.A.

E-mail address: namioka@math.washington.edu